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Optimality Criteria for a Class of Multi-Objective Nonlinear Integer Programs

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Abstract. This paper studies the Graver's optimality conditions for multi-objective non-linear integer programming problem based on Hilbert basis. Here, the result is generalized to include a fairly large class of multi-objective non-linear objective functions. This extension provides in particular a link between the superadditivity of the difference objective functions and the Hilbert bases of conic subpartitions in R^n.

Key-words: Multi-objective optimization, Graver's optimality criteria, Hilbert basis, Generalized optimality criteria.

1. Introduction

Several classes of non-linear integer programs can be successfully solved by different popular algorithms. There are branch and cut algorithms in which integrality restrictions and possibly other constraints are initially relaxed and gradually reintroduced within a branch and bound tree. Additional valid constraints can also be generated. The optimality criteria for any non-linear multi-objective linear programming problem means to derive necessary and sufficient conditions for a feasible point to be optimal with respect to some given objective functions. Genuine multi-objective optimization shows the real inter-relationships between the criteria and enables checking the correctness of the model. In one such criteria, one can systematically often devise a primal augmentation algorithm that starting with a feasible point either detects an improving direction yielding a new feasible point or terminates with a certificate that the current feasible point is optimal. Karush-Kuhn Tucker necessary and sufficient optimality conditions can be formed as a natural extension to single objective optimization for both differentiable and non-differentiable problems. In linear programming, the simplex method is an example of an algorithm that makes use of the optimality condition for basic feasible solutions by the so called reduced cost criterion.

To describe the optimality criteria in integer linear programming, some notation is required that we derive next. For a given data $A \in Z^{m \times n}$ and $b \in Z^{m}$, we investigate a multi-objective integer program of the form

$$
\begin{aligned}
&Min\ .f_1(x) \\
&Min\ .f_2(x) \\
&. \\
&. \\
&. \\
&Min\ .f_m(x) \\
&\text{s.t } x \in K
\end{aligned}
$$

where the feasible region is described as

$K = \{x \in Z_{+}^{n} : Ax = b\}$

Here Z_{+}^{n} is the set of nonnegative integers with n tuples. For a pointed rational polyhedral cone K, $H(K)$ denotes the unique Hilbert basis of K, i.e the inclusionwise minimal subset of the integer points in K such that every integer point in K can be represented as a non-negative integer combination of the elements in the set. Gordan gave a classical lemma in 1873, which gave the existence of a Hilbert basis and in 1931, Vander Corput showed that the Hilbert basis of a pointed cone K is uniquely determined. (Gordan, 1873; Vander Corput, 1931)

Resorting to the notion of Hilbert basis, we are prepared to derive optimality conditions for a multi-objective linear integer program of the form

$$
\begin{aligned}
&Min\ .C_1^T x \\
&Min\ .C_2^T x \\
&. \\
&. \\
&. \\
&Min\ .C_m^T x \\
&\text{s.t } x \in K\ .
\end{aligned}
\tag{1}
$$

Let $O_1, O_2, \ldots O_{2^n}$, denote the partition of R^n into all its orthants. Then $K_j = \{x \in R^n : Ax = 0\} \cap O_j$ is a pointed polyhedral cone in R^n for every $j \in \{1, 2, \ldots 2^n\}$. Let H_j be the unique minimal Hilbert basis of K_j .

Murota et al. (2004) provided a link between the superadditivity of the difference objective functions and the Hilbert bases of conic subpartitions in R^n for single objective linear integer programming. We extend this link for the case of multi-objective optimization.

2. **Graver's Optimality Criteria (Graver, 1975)**

A feasible point $x \in K$ for a linear integer program is optimal with respect to linear objective function vector c if and only if $C^T h \geq 0 \quad \forall h \in \bigcup_{j=1}^{2^n} H_j$ s.t $x + h \in K$

Although this optimality criteria seems to be algorithmically intractable for large numbers of n , (one would have to compute 2^n Hilbert bases) it still forms the basis of an algorithm that appears to be promising for integer programming. An exact primal augmentation algorithm was introduced for solving general linear integer programs and was proved that various versions of the algorithm are finite. (Haus, 2003) It is a major concern to show that how the sub-problem of replacing a column can be accomplished effectively. More precisely, this integral basis method solves linear integer programs based on iteratively computing Hilbert basis of discrete relaxations of the underlying integer program and reformulating the problem in a higher dimensional space. This algorithm uses many advanced techniques that are not related to these optimality criteria. In abstract mathematical terms, however, it is an integer simplex algorithm based on Hilbert bases and inspired by Graver's optimality criteria that we introduced above.

In this paper, the optimality conditions are generalized for a multi-objective integer program with linear objective functions to families of integer programs with certain nonlinear objective functions f_i . A key to obtain such a generalization is to define a chronic subpartition of the cones K_j that depends on f_i . Such partitions are refered as refined conic partition and denotd by $\{K_r(f_i)\}_r$ where r corresponds to the index assigned to each subcone in the conic subpartition.

An efficient method has been proposed for solving two related nonlinear integer programming problems arising in series-parallel reliability systems. (Sun et al., 2006) A parametric algorithm has been proposed for identifying the pareto set of a bi-objective integer program which is based on the weighted Chebyshev (Tchebycheff) scalarization, and its running time is asymptotically optimal. (Ted et al., 2006)

The paper is formulated as follows: In the next section we classify the objective functions in different ways followed by a theorem showing their property. After that we generalize the optimality criteria stated by Graver to multi-objective integer programs with nonlinear objective functions in form of two theorems. In the last section, conclusion is drawn.

3. Classification of Objective Functions

Now, we generalize the optimality criteria stated by Graver to a multi-objective integer program with nonlinear objective functions:

$$\begin{aligned} &Min\ .f_1(x)\\ &Min\ .f_2(x)\\ &.\\ &.\\ &.\\ &Min\ .f_m(x) \end{aligned} \tag{2}$$

where each $f_i : R^n \to R$ are nonlinear functions and as before

$K = \{x \in Z_+^n : Ax = b\}$ with $A \in Z^{m \times n}$ and $b \in Z^m$.

Let $\{K_j\}_j$ be the family of polyhedral cones $K_j = \{x \in R^n : Ax = 0\} \cap O_j$ where O_j is the j^{th} orthant with $j \in \{1,2,\ldots 2^n\}$. With reference to the given objective functions f_i, we consider a further partition of K_j into polyhedral cones to obtain a refined conic partition say $\{K_r(f_i)\}_r$ of $\{x \in R^n : Ax = 0\}$. By construction each $K_r(f_i)$ is contained in some K_j . The refined conic partition $\{K_r(f_i)\}_r$ will be used in expressing a local optimality criterion for $f_i \ \forall i = 1,2,\dots m$

Now we introduce three classes of objective functions F_1, F_2 and F_3 as follows:

- F_1 denotes the family of functions f_i that can be represented as

$$f_i(x) = \frac{1}{2} x^T Q_i x + d_i^T x + a_i \qquad \dots(3)$$

For some positive semidefinite symmetric matrix $Q_i \in Q_i^{n \times n}$, where $Q_i^{n \times n}$ is the set of all $n \times n$ positive semidefinite symmetric matrices, vector $d_i \in Q_i^n$ and scalar $a_i \in R$, $\forall i = 1,2,\dots m$

- F_2 denotes the family of functions f_i that can be represented as

$$f_i(x) = \sum_{q=1}^{s} \phi_{iq}\left(c_{iq}^T x\right) \qquad \dots(4)$$

For some integer s , vectors $c_{iq} \in Q_i^{n \times n}, (q = 1,2,\dots s) \ \forall i = 1,2,\dots m$, and convex functions $\phi_{iq} : R \to R, (q = 1,2,\dots s)$

- F_3 denotes the family of functions f_i that admit a refined conic partition $\{K_r(f_i)\}_r$, s.t

$$f_i(x + h_1 + h_2) + f_i(x) \geq f_i(x + h_1) + f_i(x + h_2) \qquad \dots(5)$$

For every $x \in K$ and every $h_1, h_2 \in Z^n$ with $\{h_1, h_2\} \in K_r(f_i)$ for some r and $x + h_1 + h_2 \in K$

Note that $x + h_1 + h_2 \in K$ implies that $x + h_1 \in K$ and $x + h_2 \in K$. And (5) is equivalent to $f_i(x + h_1 + h_2) - f_i(x) \geq [f_i(x + h_1) - f_i(x)] + [f_i(x + h_2) - f_i(x)]$

Which is the superadditivity of $g_{ix}(h)$ i.e

$g_{ix}(h_1 + h_2) \geq g_{ix}(h_1) + g_{ix}(h_2)$

where $g_{ix}(h) = f_i(x + h) - f_i(x)$

within a subset of each cone $K_r(f_i)$.

Theorem 1.

$$K_1 \subseteq K_2 \subseteq K_3$$

Proof:

To prove $K_1 \subseteq K_2$, let $f_i \in K_1$, $\forall i = 1,2,...m$. Then

$f_i(x) = \frac{1}{2} x^T Q_i x + d_i^T x + a_i$ for some Q_i, d_i and a_i

For any Q_i, there exists a factorization:

$Q_i = B_i D_i B_i^T$

Where $D_i = diag\ (\lambda_{i1}, \lambda_{i2},... \lambda_{in})$ with $\lambda_{i1}, \lambda_{i2},... \lambda_{in} \geq 0$ and $B_i \in Q_i^{n \times n}$ is an $n \times n$ matrix; e.g. the LDL^T factorization with pivoting gives such a factorization. (Golub, 1996).

Thus

$$f_i(x) = \frac{1}{2} \sum_{q=1}^{n} \lambda_{iq} \left(\sum_{j=1}^{n} B_{ijq} x_j \right)^2 + d_i^T x + a_i$$

$$= \sum_{q=1}^{n+1} \phi_{iq} \left(c_{iq}^T x \right)$$

With

$$\phi_{iq}(t) = \begin{cases} \frac{1}{2} \lambda_{iq} t^2 & (q = 1,2...n) \\ t + a_i & (q = n+1) \end{cases}$$

and

$$c_{iq} = \begin{cases} \left(B_{i1q}, B_{i2q}, ..., B_{inq} \right) & (q = 1,2 \ldots n) \\ \mathrm{d}_{iq} & (q = n+1) \end{cases}$$

Hence $K_1 \subseteq K_2$.

Now it remains to prove that $K_2 \subseteq K_3$.

To prove this, let $f_i \in K_2$ Then $f_i(x) = \sum_{q=1}^{s} \phi_{iq} \left(c_{iq}^T x \right)$ for some vectors c_{iq} $(q = 1,2 \ldots s)$ and convex functions ϕ_{iq} $(q = 1,2 \ldots s)$. Then we construct a refined conic partition $\{K_r(f_i)\}_r$ according to the signs of $c_{iq}^T x$ $(q = 1,2 \ldots s)$.

4. **Generalized Optimality Criteria**

Our main theorem states that the pareto optimality is guaranteed by a local optimality for objective functions in the class K_3. The local optimality is defined with reference to the refined conic partition $\{K_r(f_i)\}_r$ for a given objective function $f_i \in K_3$.

Theorem 2.

For any function $f_i \in K_3$ with the refined conic partition $\{K_r(f_i)\}_r$ and a feasible point $x^0 \in K$, the following statement holds:

x^0 is optimal if and only if

$f_i(x^0 + h) \geq f_i(x^0)$ for all $h \in \bigcup_r H(K_r(f_i))$ such that $x^0 + h \in K$.

Proof:

It is sufficient to prove the "only if" part. For all $x \in K$ there exists r such that $x - x^0 \in K_r(f_i)$. Hence $x = x^0 + \sum_{q=1}^{t} \lambda_{iq} h_q$ for some $h_q \in H(K_r(f_i))$ $(q = 1,2,\ldots, t)$ and $\lambda_{iq} \in Z_+$, $(q = 1,2,\ldots, t)$. Then

$$f_i(x) - f_i(x^0) = f_i\left(x^0 + \sum_{q=1}^{t} \lambda_{iq} h_q \right) - f_i(x^0)$$

$$\geq \sum_{q=1}^{t} \lambda_{iq} \left[f_i(x^0 + h_q) - f_i(x^0) \right] \geq 0$$

Where the first inequality is by (3) and the second is by the assumed local optimality.

The property of refining Hilbert bases does not seem to apply to arbitrary convex functions. This means that our optimality criterion does not apply to arbitrary convex functions. The above theorem can be extended for class of functions K_4 given below and the proof of this extension is obvious.

Theorem 3.

Let K_4 be the class of functions that can be obtained from some function in K_3 through a scale change of the function values. That is a function belongs to K_4 if and only if it can be represented as $f_i(x) = \phi_i g_i(x)$ with strictly increasing functions $\phi_i : R \rightarrow R$ and functions $g_i \in K_3$. Then it is easy to see that the optimality criteria in theorem 2. is valid for $f_i \in K_4$.

5. Conclusion

- Classification of objective functions is possible.
- The global optimality is guaranteed by a local optimality for objective functions in the class K_3.
- Optimality criteria in theorem 2 is valid for $f_i \in K_4$.

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